New results on correlation numbers in minimal Liouville gravity

Aleksandr Artemev [1]

[1]Landau Institute for Theoretical Physics, Chernogolovka, Russia

Minimal Liouville gravity (MLG) is a 2-dimensional model of quantum gravity. It is a CFT of total central charge 0, which consists of 3 sectors: CFT minimal model (r, r') ("matter"), "BRST"-ghost BC-system of central charge -26 and Liouville CFT, representing the dynamics of conformal factor of metric on a 2D surface. A set of interesting operators in this theory are cohomology classes of nilpotent BRST-charge $\mathcal{Q}$ that the theory has; simplest representatives of these are the so-called **"tachyon"** operators. Correlators of such operators integrated over the moduli of the surface are the so-called **"tachyon correlation numbers"**.
MLG supposedly can be equivalently described in terms of double-scaled **matrix models**. In this formulation the correlation numbers described above are much simpler to calculate. The correspondence between two formulations, however, is not yet proven in full. The most substantial progress is achieved for $(2, 2p+1)$ minimal models, on which we will concentrate in this work.
Here we discuss two recent developments connecting matrix models and CFT approaches. One concerns "first-principle" calculations of correlation numbers in higher topology (torus). The other is a proposed interpretation of matrix model answers in a certain parametric limit in terms of classical Liouville theory, or moduli space geometry.

Torus one-point correlation numbers[1]

Generalities

"Tachyon" operators are built from the product of minimal model ($\Phi_{1,n}$) and Liouville ($V_a \equiv \exp(2a\varphi)$) primary fields so that their total conformal dimension is $(1,1)$: $U_{1,n} \equiv \Phi_{1,n} V_{a_{1,-n}}$; $a_{1,-n} = b\frac{n+1}{2}, b^2 = \frac{2}{2p+1}$.
To obtain a well-defined non-zero correlation number, one should additionaly dress these with BRST-ghosts (number of which depends on the studied topology) and integrate over moduli space. For the simplest (one-point) correlator on the torus it is defined as follows:

$$\int_{\mathcal{F}} d^2\tau \, \langle B\overline{B}C\overline{C}V_{1,-n}\Phi_{1,n}\rangle_\tau \tag{1}$$

One can show that this correlator is modular invariant. "Brute force" calculation is difficult, because it involves integration of not explicitly known functions (generic Liouville conformal blocks) over intermediate Liouville dimension

$$\langle V_a\rangle_\tau = \int_\gamma \frac{dP}{4\pi} C^{(L)Q/2+iP}_{a,Q/2+iP} (q\overline{q})^{-1/24+P^2} \times |F_L(\Delta^{\mathsf{L}}_a, \Delta^{\mathsf{L}}_{Q/2+iP}, q)|^2, \; q \equiv \exp(2\pi i\tau) \; , \tag{2}$$

and moduli τ.

Higher equations of motion

Progress in previous analytic calculations of correlation numbers in sphere topology [2] was achieved thanks to **"higher equations of motion"** (HEM), discovered by Al.Zamolodchikov [3]. We will use the following form of these equations

$$C\overline{C}V_{1,-n}\Phi_{1,n} = B^{-1}_{1,n}\mathcal{Q}\overline{\mathcal{Q}}\left(O'_{1,n}\right), \; O'_{1,n} \equiv H_{m,n}\overline{H}_{m,n}V'_{1,n}\Phi_{1,n} \tag{3}$$

The prime signifies derivative wrt to Lioville parameter a, $H_{1,n}$ are operators built of Virasoro and ghost modes; e.g. $H_{1,2} = M_{-1} - L_{-1} + b^2CB$. Operators $O'_{1,n}$ in the RHS are built from **degenerate** Liouville and MM operators; that simplifies the OPE. In fact, $O_{1,n} \equiv H_{m,n}\overline{H}_{m,n}V_{1,n}\Phi_{1,n}$ — the so-called "ground ring" operators — represent non-trivial $\mathcal{Q}$-cohomology classes of ghost number zero and are known to have very simple product with tachyons (in cohomology). Moreover, it turns out that using these equations allows to calculate the moduli integral over τ very easily.

Reduction to boundary terms

If one uses HEM in the form before, up to $\mathcal{Q}, \overline{\mathcal{Q}}$-exact terms (which average to zero) one-point tachyon correlator can be rewritten as

$$\langle T(z)\overline{T}(\overline{z})H_{1,n}\overline{H}_{1,n}V'_{1,n}\Phi_{1,n}\rangle \tag{4}$$

(since $\{\mathcal{Q}, B(z)\} = T(z)$). One can use **conformal Ward identities** to simplify this expression. On the torus, Ward identities include derivative with respect to the moduli τ; e.g. for one insertion of T

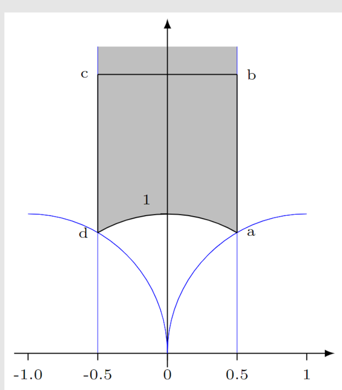
$$\langle T(z)\Phi_\Delta(x)\rangle = \left[\Delta\left(\mathcal{P}(z-x) + 2\eta_1\right) + (\zeta(z-x) + 2\eta_1 x)\partial_x + 2\pi i\frac{\partial}{\partial\tau}\right]\langle\Phi_\Delta(x)\rangle \tag{5}$$

The explicit calculation can only be done knowing explicit expressions for $H_{m,n}$, but, based on the study of different cases, one can conjecture that it always reduces to

$$\frac{1}{2}\frac{\partial}{\partial a}\langle T(z)\overline{T}(\overline{z})H_{1,n}\overline{H}_{1,n}V_a\Phi_{1,n}\rangle \, |_{a=a_{1,n}} = (2\pi)^2\frac{\partial^2}{\partial\tau\partial\overline{\tau}}\langle O'_{1,n}\rangle \tag{6}$$

Thus, moduli integral can successfully be reduced to contributions from the boundary of moduli space, using Gauss theorem $\int_F dS\left(\vec{\nabla}, \vec{A}\right) = \int_{\partial F} dl\,(\vec{n}, \vec{A})$. A particular fact that should be proven to verify this is that $\langle O_{1,n}\rangle_\tau$ is independent of modular parameter τ; this is a consequence of their $\mathcal{Q}$-closedness.

Moduli space $\mathcal{F}$ can be represented as fundamental domain of $SL(2,\mathbb{Z})$ action on upper half-plane; one of the choices looks as pictured, having 4 boundary segments



Contributions from cd and ab cancel each other, because correlator is periodic in Re τ with period 1. To calculate integral over cb at Im $\tau \to \infty$ we only need to consider leading asymptotics for conformal block series in q, which are easily evaluated. Finally, two arcs $1d$ and $1a$ are glued with modular transformation $\tau \to -1/\tau$, so integral over them doesn't cancel only due to **non-covariance** of $\langle O'_{1,n}\rangle_\tau$. From **modular transformation property**

$$\langle O'_{1,n}\rangle_{-1/\tau} = \langle O'_{1,n}\rangle_\tau + \frac{1}{2}\frac{\partial}{\partial a}\Delta^{\mathsf{L}}_a \, |_{a=a_{1,n}} \cdot \log(\tau\overline{\tau})\langle O_{1,n}\rangle_\tau \tag{7}$$

it can be reduced to $\frac{1}{24}(2\pi)^3 \cdot \frac{1}{2}\frac{\partial}{\partial a}\Delta^{\mathsf{L}}_a \cdot \langle O_{1,n}\rangle$ (with $\langle O_{1,n}\rangle$ independent of τ), that is also easily evaluated considering the asymptotic Im $\tau \to \infty$.

Final answer

Evaluation of the remaining one-point correlators at Im $\tau \to \infty$ is straightforward enough; up to normalization of operators (which is explicitly known and conforming with spherical case) **the answer is**

$$\int d^2\tau \, \langle B\overline{B}C\overline{C}U_{1,k}\rangle \sim k(2p+1-k) \tag{8}$$

which **coincides** with already known result from matrix models [4].

Semiclassical limit of tachyon correlators and moduli space volumes[5],[6]

Some properties of correlation numbers in semiclassical limit

In general case, correlation numbers in $(2, 2p+1)$ MLG are non-analytic functions (in certain normalization — piecewise-defined polynomials) of p and k_i. E.g. for the sphere 4-point correlator of tachyons U_{1,k_i+1} with $0 \le k_1 \le k_2 \le k_3 \le k_4 \le p-1$ we have [7]

$$Z_{k_1k_2k_3k_4} = -F_\theta(-2) + \sum_{i=1}^4 F_\theta(k_i - 1) - F_\theta(k_{12|34}) - F_\theta(k_{13|24}) - F_\theta(k_{14|23}) \tag{9}$$

where $k_{ij|lm} = \min(k_i + k_j, k_l + k_m)$; $\quad F_\theta(k) = \frac{1}{2}(p-k-1)(p-k-2)\theta(p-k-2)$. Some properties of these expressions were found which are especially interesting in the "semiclassical" limit $p \to \infty$, $\kappa_i \equiv b^2(1+k_i)$ finite.

1. When $k_i + k_j > p \;\; \forall i,j$, it reduces to a symmetric polynomial

$$-\frac{4\pi^2}{b^4}Z_{k_1k_2k_3k_4} = 2\pi^2 + \frac{6\pi^2}{(2p+1)^2} - \frac{4\pi^2}{2}\sum_{i=1}^4(1-\kappa_i)^2 \tag{10}$$

   Without $1/p^2$ corrections, it coincides with **Weil-Petersson** moduli space volumes of hyperbolic surfaces with geodesic boundaries, analytically continued to imaginary lengths $l_i = 2\pi i(1-\kappa_i)$.
2. When $k_i + k_j < p \;\; \forall i,j$, the correlators on the sphere **"count conformal blocks"** in the minimal model sector. I.e. the correlator vanishes if the so-called "fusion rules" $k_1 + k_2 + k_3 > k_4$ are not satisfied and, if nonzero, is equal to $(k \equiv \sum k_i)$

$$Z_{k_1k_2k_3k_4} = \begin{cases}(1+k_1)(2p-3-k), \; k_{14} < k_{23}\\ (1 + \frac{k_2+k_3+k_1-k_4}{2})(2p-3-k), \; k_{14} \ge k_{23}\end{cases} \tag{11}$$

Interpretation as "moduli space volumes"

It was proposed to interpret classical limit of MLG correlators as **"moduli space volumes for constant curvature surfaces with conical defects"**. This interpretation allows to understand some the properties above: e.g. "Weil-Petersson" property is understood from the fact that conical defect can be obtained from analytically continuing metric in the vicinity of the geodesics

$$ds^2 = \frac{dt^2}{t^2+\epsilon^2} + (t^2+\epsilon^2)d\alpha^2 \to (\epsilon = i\tilde{\epsilon}, t = \tilde{\epsilon}\cosh r) \to ds^2 = dr^2 + \tilde{\epsilon}^2\sinh^2 r\, d\alpha^2 \tag{12}$$

This analytic continuation, however, fails for "blunt" defects, when requirement $\kappa_i + \kappa_j > 1 \; \forall i,j$ is not satisfied. The "fusion rules" in the form $\kappa_1 + \kappa_2 + \kappa_3 > \kappa_4$ are known to be an existence condition for constant (positive) curvature metrics **(the so-called Troyanov region)**; .

Connection with classical Liouville theory

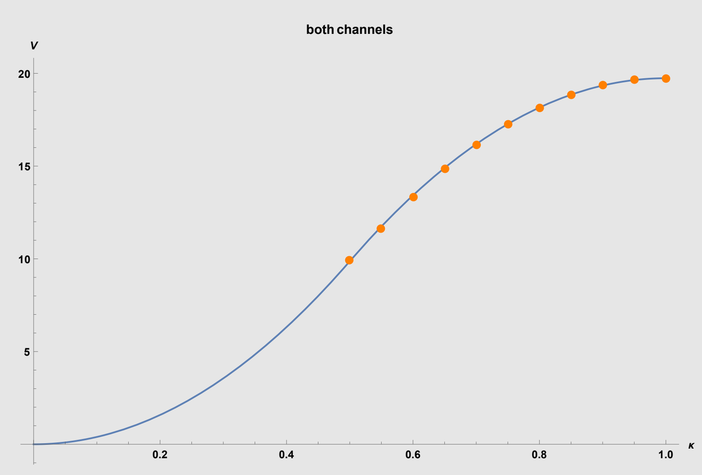
A mathematical definition of these volumes in terms of intersection theory on moduli space, that accounts for their non-analyticity, was only proposed recently [8]. However, apparently they are related to a quite old result of Takhtajan and Zograf [9]. They proposed to treat (properly regularized) **classical action** of Liouville theory as **Kahler potential** for certain metrics on moduli space of surfaces with marked points. Properties of the volumes associated with these metrics agree with aforementioned ones of MLG correlators.

Check 1: numeric calculation from CFT

Liouville 4-point correlator is decomposed as

$$\langle V_{a_1}(0)V_{a_2}(x)V_{a_3}(1)V_{a_4}(\infty)\rangle = \int\frac{dP}{4\pi}C(a_1, a_2, \frac{Q}{2} - iP)\,C(\frac{Q}{2} + iP, a_3, a_4)\left|F_\Delta\left(\begin{array}{cc}\Delta_1 & \Delta_3\\ \Delta_2 & \Delta_4\end{array}\middle| x\right)\right|^2 \tag{13}$$

Liouville parameters $a_i \approx \kappa_i b^{-1}$. In $b \to 0$ limit structure constants and conformal blocks exponentiate; asymptotics of both are known from conformal field theory. The integrand becomes of the form $\sim \exp\left(-\frac{1}{b^2}S_{cl}(p, x, \overline{x})\right)$. When all defects are "sharp", there is a saddle point with real intermediate Liouville momentum $p_{\text{saddle}} : \frac{\partial S_{cl}}{\partial p}\,|_{p=p_{\text{saddle}}} = 0$ which gives a leading approximation to the integral. One can find a saddle point p_{saddle} as a series expansion on the boundary of moduli space $(x \to 0)$, calculate S_{cl} as a similar series and then — the corresponding Kahler metric $g_{x\overline{x}} = -4\pi\partial\overline{\partial}S_{cl}$ and the volume $\int d^2x\sqrt{\det g}$. For an explicit example with $\kappa_i = \{\kappa, \kappa, 1, 1\}$ numerical results coincide with MLG answer (to the order of the expansion for small x that we considered, the deviation is of order 1%)



Check 2: perturbation theory for monodromy method

Consider a situation when $\kappa_1 \ll 1$ and all defects are "blunt". Then, expression for classical limit of four-point correlation number should reduce to

$$Z_{\kappa_1\kappa_2\kappa_3\kappa_4} \sim \kappa_1(2-\kappa) \tag{14}$$

It is known that to calculate the classical Liouville action, one can consider instead a linear **Fuchsian equation** $(2\delta_i \equiv \kappa_i(1-\kappa_i/2))$

$$[\partial_z^2 + t(z)]\psi = 0, \; t(z) = \frac{\delta_1}{(z-x)^2} + \frac{\delta_2}{z^2} + \frac{\delta_3}{(z-1)^2} + \frac{x(x-1)c}{z(z-1)(z-x)} + \frac{\delta_4 - \delta_3 - \delta_2 - \delta_1}{z(z-1)} \tag{15}$$

and find "accessory parameter" $c(x, \overline{x})$ such that monodromy group of its solutions is isomorphic to either $SU(2)$ or $SU(1,1)$ (depending on the sign of curvature). Then classical action and Zograf-Takhtajan metric are determined from "Polyakov formula" $c = -\frac{\partial S_{cl}}{\partial x}$. In the considered limit this problem can be solved in perturbation theory in κ_1; the result

$$Z_{\kappa_1\kappa_2\kappa_3\kappa_4} \sim \kappa_1(2 - \kappa_2 - \kappa_3 - \kappa_4) \tag{16}$$

to first order agrees with MLG correlator. Analogous expression can be found for the case $\kappa_1, \kappa_2 \ll 1$, exhibiting expected non-analytic structure and being proportional to min (κ_1, κ_2).

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[1] A. Artemev and V. Belavin. *Torus one-point correlation numbers in minimal Liouville gravity.* arXiv: 2210.14568 [hep-th].

[2] A. Belavin and A. Zamolodchikov. *Integrals over moduli spaces, ground ring, and four-point function in minimal Liouville gravity.* [Teor. Mat. Fiz.147,339(2006)]. 2006. DOI: 10.1007/s11232-006-0075-8.

[3] A. Zamolodchikov. *Higher equations of motion in Liouville field theory.* arXiv: hep-th/0312279 [hep-th].

[4] A. Belavin and G. Tarnopolsky. *Two dimensional gravity in genus one in Matrix Models, Topological and Liouville approaches.* arXiv: 1006.2056 [hep-th].

[5] A. Artemev. *Note on large-p limit of (2,2p+1) minimal Liouville gravity and moduli space volumes.* arXiv: 2203.06629 [hep-th].

[6] A. Artemev. *$p \to \infty$ limit of tachyon correlators in $(2, 2p+1)$ minimal Liouville gravity from classical Liouville theory.* arXiv: 2305.08118 [hep-th].

[7] A. Belavin and A. Zamolodchikov. *On Correlation Numbers in 2D Minimal Gravity and Matrix Models.* arXiv: 0811.0450 [hep-th].

[8] L. Eberhardt and G. J. Turiaci. *2D dilaton gravity and the Weil-Petersson volumes with conical defects.* arXiv: 2304.14948 [hep-th].

[9] L. Takhtajan and P. Zograf. *Hyperbolic 2-spheres with conical singularities, accessory parameters and Kaehler metrics on $\mathcal{M}_{0,n}$.* arXiv: math/0112170 [math.CV].